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The question of mass in (anti-) de Sitter spacetimes

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Abstract

The possible existence of a non-zero cosmological constant Λ gives rise to controversial interpretations. By Λ we here understand some sort of bare cosmological constant, and not the observed one that should contain modifications coming from the classical or the quantum fluctuations of matter fields. Is Λ a universal constant fixing the geometry of an empty universe, as fundamental as the Planck constant or the speed of light in the vacuum? Is it instead something emerging from a perturbative calculus performed on the metric solution of the Einstein equation and to which it might be given a material status of (dark or bright) 'energy'? Since a physical quantity like mass originates in a Minkowskian conservation law, we proceed to a group theoretical interpretation of this relation in terms of the two possible Λ-deformations of the Poincaré group, namely the de Sitter and anti de Sitter groups. We use the so-called Garidi mass in order to make clear the asymptotic relations between Minkowskian masses m and their possible dS/AdS counterparts.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction: the question of mass, a matter of debate

1.1. The mass in flat spacetime

Let us start with some quotations excerpted from current literature:

Okun: [1] It is firmly established that all particles of a given kind (. . .) are identical and hence have exactly the same value of mass. The same refers to protons and neutrons.

Yao *et al*: [2] The ratios m_u/m_d and m_s/m_d are extracted from pion and kaon masses using chiral symmetry. The estimates of u and d masses are not without controversy and remain under active investigation. Within the literature there are even suggestions that the u quark could be essentially massless

Wilczek: [3] [From QCD] we find that 90% of the proton (and neutron) mass, and therefore 90% of the mass of ordinary matter, emerges from an idealized theory whose ingredients are entirely massless

Wilczek: [3] A major goal of theoretical physics is to describe the world with the smallest number of concepts. For that reason alone, it is an important result that we can largely eliminate mass as a property necessary to describe matter

Wilczek: [3] We continue to search for concepts and theories that will allow us to understand the origin of mass in all its forms, by unveiling more of Nature's hidden symmetries

It appears clear from these few sentences that the concept of mass is far from reaching a consensus in the physics community! However, it is commonly taken as granted that in flat Minkowski spacetime, the concept of (rest) mass originates in the ubiquitous law of conservation of energy, a direct consequence of the Poincaré symmetry and the hypothesis of the existence of elementary systems in Nature (in an asymptotic sense). It is always worthy to recall the point of view of Wigner [4] on this question:

(i) *The concept of an 'elementary system' requires that all states of the system be obtainable from the relativistic transforms of any state by superpositions. In other words, there must be no relativistically invariant distinction between the various states of the system which would allow for the principle of superposition. This condition is often referred to as irreducibility condition*

(ii) *The concept of an elementary system (. . .) is a description of a set of states which forms, in mathematical language, an irreducible representation space for the inhomogeneous Lorentz (≃Poincaré) group*

At this point, it is useful to recall the group theoretical arguments backing a definition of mass based on spacetime symmetry. These arguments rest upon the Wigner classification of the Poincaré UIR's [5]: the UIR's of the Poincaré group are completely characterized by the eigenvalues of its two Casimir operators, the quadratic $Q^{(1)} = P^\mu P_\mu = P^{0^2} - \mathbf{P}^2$ (Klein–Gordon operator) with eigenvalues $\langle Q^{(1)}\rangle = m^2c^2$, and the quartic $Q^{(2)} = W^\mu W_\mu$, $W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma$ (Pauli–Lubanski operator) with eigenvalues (in the non-zero mass case) $\langle Q^{(2)}\rangle = -m^2c^2 s(s+1)\hbar^2$.

From the Wigner classification of the Poincaré UIR's according to the mass operator and the little group UIR's recalled in table 1, it is commonly accepted that the only physical cases are, respectively,

(a) massive representations with positive energy, denoted as $\mathcal{P}^>(m, s)$,
(c) massless representations with positive energy, denoted as $\mathcal{P}^>(0, s)$,
(f) vacuum.

1.2. The mass in a curved background

In a curved background, the mass of a test particle can always be considered as the rest mass of the particle as it should locally hold in a tangent Minkowskian spacetime. However, when we deal with a de Sitter or anti de Sitter background, which are constant curvature spacetimes, another way to examine this concept of mass is possible and should also be considered. It

Table 1. Wigner classification of the Poincaré UIR's according to the mass operator and the little group UIR's.

First Casimir or squared mass	k^μ	Little group
(a) $P^2 = m^2c^2 > 0$, $P^0 > 0$	$(mc, 0, 0, 0)$	$SO(3)$
(b) $P^2 = m^2c^2 > 0$, $P^0 < 0$	$(-mc, 0, 0, 0)$	$SO(3)$
(c) $P^2 = 0$, $P^0 > 0$	$(\kappa, \kappa, 0, 0)$	$ISO(2)$
(d) $P^2 = 0$, $P^0 < 0$	$(-\kappa, \kappa, 0, 0)$	$ISO(2)$
(e) $P^2 = N^2 > 0$	$(0, N, 0, 0)$	$SO(2, 1)$
(f) $P^\mu = 0$	$(0, 0, 0, 0)$	$SO(3, 1)$

is precisely based on symmetry considerations in the above Wigner sense, i.e. based on the existence of the de Sitter or anti de Sitter groups that are both one-parameter deformations of the Poincaré group. In particular, within this interpretative framework, one may expect to lose a precise distinction between 'massive' and 'massless'. Thus, we should look for other properties, e.g. existence or violation of conformal invariance, of some gauge invariance, in view of extending concepts about mass inherited from Minkowskian physics. Attributing a non-zero mass or a null mass to dS/AdS elementary systems might depend on the fundamental nature of spacetime.

Roughly speaking, there exist in the physics community two points of view, already present in the reflections by Einstein while dealing respectively with local gravitational phenomena and within a cosmological context. The first one is based on the following equation:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\kappa T_{\mu\nu}. \tag{1}$$

Here, the fundamental state that contains the maximum number of symmetries is the Minkowskian geometry. The second one involves explicitly Λ as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu}. \tag{2}$$

Here, the fundamental state that contains the maximum number of symmetries is the de-Sitter (dS)/anti-de-Sitter (AdS) geometry.

Since the beginning of the eighties, the de Sitter space has been considered as a key model in the inflationary cosmological scenario where it is assumed that the cosmic dynamics was dominated by a term acting like a cosmological constant. More recently, observations on far high redshift supernovae, galaxy clusters and cosmic microwave background radiation (see, for instance, [6]), suggest an accelerating universe. Again, this can be explained in a satisfactory way with such a term. This current 'inflation' is based on (increasingly reliable) current observations. The other one is of a totally dynamical nature and is still subject to controversies. This can be summarized in decomposing Λ into Λ_{bare} and Λ_{vacuum}, i.e., into a bare cosmological background and an extra term which is of quantum origin, the latter assuming large enough values for justifying inflation scenario. Also, it is obvious that Λ is not thought as the unique responsible for the complete history of the growth of the universe. Other matter entities ($\rho_{\text{matter}}, \rho_{\text{rad}}, \ldots$) are important in different epochs.

In this contribution, we intend to explain at length what could be the consequences of having a non-null Λ, whatever its origin, on our way of considering masses (see, for instance, [7] for a discussion about the massive or massless graviton). In section 2, we give a short compendium of motivations for studying in an extensive fashion quantum physics in a dS/AdS arena. Section 3 is a review of the dS/AdS geometries and classical and quantum symmetries encoded in a list of UIR's of the corresponding kinematical groups. In section 4, we consider the physical content of dS/AdS theories from a local Minkowskian point of view. The involved mathematics pertains to the contraction techniques for UIR's and the definition of the Garidi mass (for dS) and of its AdS counterpart result from this analysis. Section 5 is a reflection about the asymptotic meaning of the dS/AdS UIR parameters and their relations to the Minkowskian concept of mass. Section 6 is a short conclusion.

A large part of the presented material here has been known for a long time, and discussed in many places. However, it seemed to us necessary to give a comprehensive review of the mathematical results concerning the dS/AdS UIR's and their contraction limits in order to present an interesting mass formula for dS/AdS. This formula has been recently proposed by Garidi [8] for the de Sitter relativity. We will present a similar formula for the anti de Sitter relativity and will discuss about the relevance of such formulae in the general debate about mass we just mentioned in the beginning of this section.

2. De Sitter and anti de Sitter spacetimes as possible backgrounds

De Sitter and anti de Sitter spacetimes are, with Minkowski, the only maximally symmetric spacetime solutions in general relativity. Their respective invariance (in the relativity or kinematical sense) groups are the ten-parameter deSitter$\mathrm{SO}_0(1,4)$ and anti deSitter $SO_0(2,3)$ groups. Both may be seen as deformations of the proper orthochronous Poincaré group $\mathbb{R}^{1,3} \rtimes SO_0(1,3)$, the kinematical group of Minkowski.

As recalled above, the de Sitter (resp. anti de Sitter) spacetimes are the unique maximally symmetric solutions of the vacuum Einstein's equations with positive (resp. negative) cosmological constant Λ. This constant is linked to the (constant) Ricci curvature 4Λ of these spacetimes and it allows us to introduce the fundamental curvature or the inverse length $\varkappa = Hc = \sqrt{|\Lambda|/3}$ (H is the Hubble constant).

On the fundamental level, matter and energy are of quantum nature. But the usual quantum field theory is designed in the Minkowski spacetime. Most of the theoretical and observational [6] arguments privileging a de Sitter-like universe plead in favor of setting up a rigorous quantum field theory in de Sitter spacetime, or at least exploring specific features which could show up in such a framework and which would not have any counterpart in the flat curvature limit. Fortunately, the symmetry properties of dS universes may allow the construction of such a theory. For recent works on this subject, see for instance [9] and references therein. We should also note that the study of de Sitter spacetime offers a specific interest because of the regularization opportunity afforded by the curvature parameter as a 'natural' cutoff for infrared or other divergences. On the other hand, we should be also aware that some of our most familiar concepts like time (see, for instance, the ambiguity in choosing the static coordinate time versus the conformal time), rest mass, energy, momentum, etc, disappear, or at least need radical modifications in de Sitterian relativity, as we will comment later on.

With a given (rest Minkowskian) mass m and with the existence of a non-zero curvature is naturally associated the typical dimensionless parameter for dS/AdS perturbation of the Minkowskian background:

$$\vartheta \equiv \vartheta_m =: \frac{\hbar\sqrt{|\Lambda|}}{\sqrt{3}mc} = \frac{\hbar H}{mc^2}. \tag{3}$$

We give in table 2 the values assumed by the quantity ϑ when m is taken as some known masses and Λ (or H_0) is given its present-day estimated value. We easily understand from this table that the currently estimated value of the cosmological constant has no practical effect on our familiar massive fermion or boson fields. Contrariwise, adopting the de Sitter point

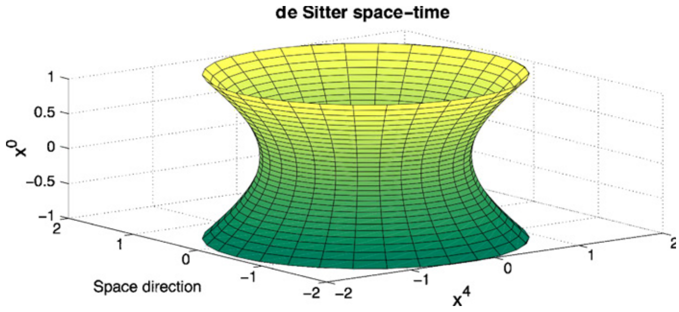

Figure 1. de Sitter spacetime as a hyperboloid embedded in $\mathbb{R}^5$.

Table 2. Estimated values of the dimensionless physical quantity $\vartheta \equiv \vartheta_m =: \frac{\hbar\sqrt{|\Lambda|}}{\sqrt{3}mc} = \frac{\hbar H}{mc^2} \approx 0.293 \times 10^{-68} \times m_{\mathrm{kg}}^{-1}$ for some known masses m and the present-day estimated value of the Hubble length $c/H_0 \approx 1.2 \times 10^{26}$m [10].

Mass m	$\vartheta_m \approx$
$m_\Lambda/\sqrt{3} \approx 0.293 \times 10^{-68}$ kg	1
Up. lim. photon mass m_γ	0.29×10^{-16}
Up. lim. neutrino mass m_ν	0.165×10^{-32}
Electron mass m_e	0.3×10^{-37}
Proton mass m_p	0.17×10^{-41}
$W^\pm$ boson mass	0.2×10^{-43}
Planck mass M_{Pl}	0.135×10^{-60}

of view appears as inescapable when we deal with infinitely small masses, as is done in the standard inflation scenario.

3. (Anti-) de Sitter geometries and (quantum) kinematics

3.1. Hyperboloids

de Sitter geometry. The corresponding de Sitter space is conveniently described as a one-sheeted hyperboloid embedded in a five-dimensional Minkowski space (the bulk):

$$M_{\mathrm{dS}} \equiv \{x \in \mathbb{R}^5; x^2 = \eta_{\alpha\beta} x^\alpha x^\beta = -\varkappa^{-2}\}, \qquad \alpha, \beta = 0, 1, 2, 3, 4, \tag{4}$$

where $\eta_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1, -1)$.

We can introduce for instance the global coordinates $t \in \mathbb{R}, \vec{n} \in S^2, \alpha \in [0, \pi]$:

$$x = \begin{cases} x^0 = \varkappa^{-1} \sinh(\varkappa ct) \\ \vec{x} = \varkappa^{-1} \cosh(\varkappa ct) \sin(\varkappa r)\vec{n} \\ x^4 = \varkappa^{-1} \cosh(\varkappa ct) \cos(\varkappa r). \end{cases} \tag{5}$$

The de Sitter spacetime is shown in figure 1.

Anti de Sitter geometry. The anti de Sitter space can be viewed as a one-sheeted hyperboloid embedded in another five-dimensional space with different metrics:

$$M_{\mathrm{dS}} \equiv \{x \in \mathbb{R}^5; x^2 = \eta_{\alpha\beta} x^\alpha x^\beta = -\varkappa^{-2}\}, \qquad \alpha, \beta = 0, 1, 2, 3, 5, . \tag{6}$$

where $\eta_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1, 1)$.

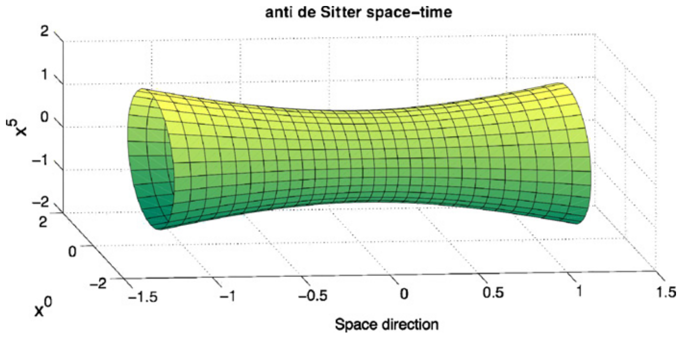

Figure 2. Anti de Sitter spacetime as a hyperboloid embedded in $\mathbb{R}^5$.

Global coordinates $t \in [0, 2\pi)$, $r \in [0, \infty)$, $\vec{n} \in S^2$ are then defined by

$$x = \begin{cases} x^0 = \varkappa^{-1} \cosh(\varkappa r) \sin(\varkappa ct) \\ \vec{x} = \varkappa^{-1} \sinh(\varkappa r)\vec{n} \\ x^5 = \varkappa^{-1} \cosh(\varkappa r) \cos(\varkappa ct) \end{cases} \tag{7}$$

The anti de Sitter spacetime is shown in figure 2.

3.2. The de Sitter group, its unitary irreducible representations, and their physical interpretation

The de Sitter relativity group is $G = SO_0(1, 4)$, i.e. the component connected to the identity of the ten-dimensional pseudo-orthogonal group $SO(1, 4)$. A familiar realization of the Lie algebra is that one generated by the ten Killing vectors:

$$K_{\alpha\beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha. \tag{8}$$

It is worthy to note that there is no globally time-like Killing vector in de Sitter, the adjective time-like (resp. spacelike) referring to the Lorentzian four-dimensional metric induced by that of the bulk. The universal covering of the de Sitter group is the symplectic $Sp(2, 2)$ group, which is needed when dealing with half-integer spins.

Specific quantization procedures applied to classical phase spaces viewed as co-adjoint orbits of the group lead to their quantum counterparts, namely the quantum elementary systems associated in a biunivocal way to the UIR's of the de Sitter group $Sp(2, 2)$. Let us give a complete classification of the latter, following the work by Dixmier [11]. We recall that the ten Killing vectors (8) can be represented as (essentially) self-adjoint operators in Hilbert space of (spinor-)tensor-valued functions on M_{dS}, square integrable with respect to some invariant inner product, more precisely of the Klein–Gordon type. These operators take the form

$$K_{\alpha\beta} \longrightarrow L_{\alpha\beta} = M_{\alpha\beta} + S_{\alpha\beta}, \tag{9}$$

where the orbital part is $M_{\alpha\beta} = -\mathrm{i}(x_\alpha \partial_\beta - x_\beta \partial_\alpha)$ and the spinorial part $S_{\alpha\beta}$ acts on the indices of functions in a certain permutational way. Like for the UIR of the Poincaré group, there are two Casimir operators, the eigenvalues of which determine completely the UIR's. They respectively read:

$$Q^{(1)} = -\tfrac{1}{2} L_{\alpha\beta} L^{\alpha\beta}, \tag{10}$$

with eigenvalues

$$\langle Q^{(1)} \rangle_{\mathrm{dS}} = -p(p + 1) - (q + 1)(q - 2), \tag{11}$$

and

$$Q^{(2)} = -W_\alpha W^\alpha, \qquad W_\alpha = -\tfrac{1}{8}\epsilon_{\alpha\beta\gamma\delta\eta}L^{\beta\gamma}L^{\delta\eta}, \tag{12}$$

with eigenvalues

$$\langle Q^{(2)}\rangle_{\mathrm{dS}} = -p(p+1)q(q-1). \tag{13}$$

Therefore, one must distinguish between

- *the discrete series* $\Pi^{\pm}_{p,q}$,
  defined by p and q having integer or half-integer values, $p \geqslant q$, q having a spin meaning. Here, we must again distinguish between
  - *the scalar case* $\Pi_{p,0}$, $p = 1, 2, \ldots$;
  - *the spinorial case* $\Pi^{\pm}_{p,q}$, $q > 0$, $p = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots, q = p, p-1, \ldots, 1$ or $\frac{1}{2}$
- *The principal and complementary series* $\Upsilon_{p,\sigma}$,
  where p has a spin meaning. We put $\sigma = q(1-q)$ which gives $q = \frac{1}{2}(1 \pm \sqrt{1-4\sigma^2})$. Like in the above, one distinguishes between
  - *The scalar case* $\Upsilon_{0,\sigma}$, where
    - $-2 < \sigma < \frac{1}{4}$ for the complementary series;
    - $\frac{1}{4} \leqslant \sigma$ for the principal series.
  - *The spinorial case* $\Upsilon_{p,\sigma}$, $p > 0$, where
    - $0 < \sigma < \frac{1}{4}$, $p = 1, 2, \ldots$, for the complementary series,
    - $\frac{1}{4} \leqslant \sigma$, $p = 1, 2, \ldots$, for the integer spin principal series,
    - $\frac{1}{4} < \sigma$, $p = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, for the half-integer spin principal series.

3.3. The anti de Sitter group, its unitary irreducible representations, and their physical interpretation

The anti de Sitter relativity group is $G = SO_0(2,3)$. Like for dS, a realization of the Lie algebra is that one generated by the ten Killing vectors:

$$K_{\alpha\beta} = x_\alpha\partial_\beta - x_\beta\partial_\alpha. \tag{14}$$

Contrarily to dS, there is one globally time-like Killing vector in anti de Sitter, namely K_{50}. On the other hand, the compact nature of the associated one-parameter group (it is just $SO(2) \simeq U(1)$ or its double covering) can raise problems [12]. The latter can be circumvented by dealing with the universal covering $\widetilde{G} = \widetilde{SO_0(2,3)}$ in which the 'time' $SO(2)$ subgroup becomes $\mathbb{R}$. The two-covering of the anti de Sitter group is the symplectic $Sp(4,\mathbb{R})$ group, which is needed when dealing with half-integer spins. Here, the UIR's of the de anti de Sitter group $Sp(4,\mathbb{R})$ which are physically meaningful are found in the holomorphic discrete series and in its lower limits. Like in dS, the infinitesimal generators read as $K_{\alpha\beta} \longrightarrow L_{\alpha\beta} = M_{\alpha\beta} + S_{\alpha\beta}$.

In the case of the discrete series and its lower limit, these UIR's are denoted as $D(\varsigma, s)$ with $2s \in \mathbb{N}$ and $\varsigma \geqslant s+1$ (at the exception of a few cases). The label s is for spin (it plays the role of the dS p) and ς for lowest 'energy' (to some extent it plays the role of the dS q). For UIR in the strictu senso discrete series of $Sp(4,\mathbb{R})$, the parameter ς is such that $2\varsigma \in \mathbb{N}$ whilst for 'discrete' series UIR of the universal covering $\widetilde{SO_0(2,3)}$ this parameter assumes its values in $[s+1, \infty)$. Here too, there are two Casimir operators, the eigenvalues of which determine completely the UIR's. With our parameters, they read as

$$Q^{(1)} = -\tfrac{1}{2}L_{\alpha\beta}L^{\alpha\beta}, \tag{15}$$

with eigenvalues

$$\langle Q^{(1)}\rangle_{\mathrm{AdS}} = s(s+1) + \varsigma(\varsigma-3), \tag{16}$$

and

$$Q^{(2)} = -W_\alpha W^\alpha, \qquad W_\alpha = -\tfrac{1}{8}\epsilon_{\alpha\beta\gamma\delta\eta}L^{\beta\gamma}L^{\delta\eta}, \tag{17}$$

with eigenvalues

$$\langle Q^{(2)}\rangle_{\mathrm{AdS}} = -s(s+1)(\varsigma-1)(\varsigma-2). \tag{18}$$

Among the AdS UIR $D(\varsigma, s)$, one must distinguish between those for which $\varsigma > s+1$, and the following important limit cases

- *The limit scalar cases* $D(1,0)$ and $D\left(\frac{1}{2},0\right)$. The latter is called the 'Rac' [13].
- *The limit spinorial or tensorial cases* $D(s+1,s)$ and $D\left(1,\frac{1}{2}\right)$. The latter is called the 'Di' [13].

4. Minkowskian content of dS and AdS elementary systems: contraction results and the Garidi mass

Now, we wish to go further into the interpretative problem of mass in a dS/AdS background. The crucial question to be addressed concerns the interpretation of the dS/AdS UIR's (or quantum AdS and dS elementary systems) from a (asymptotically) Minkowskian point of view. We mean by this the study of the contraction limit $\varkappa \to 0$ or equivalently $\Lambda \to 0$ of these representations, which is the quantum counterpart of the following geometrical and group contractions.

Flat limit of de Sitter geometry

- $\lim_{\varkappa\to 0} M_{\mathrm{dS}} = M_0$, the Minkowski spacetime tangent to M_{dS} at, say, the de Sitter origin point $O_{\mathrm{dS}} = (0, \vec{0}, \varkappa^{-1})$, since then $M_{\mathrm{dS}} \ni x \approx_{\varkappa\to 0} (t, \vec{r} = r\vec{n}, \varkappa^{-1})$ from equation (5).
- $\lim_{\varkappa\to 0} Sp(2,2) = \mathcal{P}_+^\uparrow(1,3) = M_0 \rtimes SL(2,\mathbb{C})$, the Poincaré group.

As a matter of fact, the ten de Sitter Killing vectors (8) contract to their Poincaré counterparts $K_{\mu\nu}, \Pi_\mu, \mu = 0,1,2,3$, after rescaling the four $K_{4\mu} \longrightarrow \Pi_\mu = \varkappa K_{4\mu}$.

Flat limit of anti de Sitter geometry

- $\lim_{\varkappa\to 0} M_{AdS} = M_0$, the Minkowski spacetime tangent to M_{AdS} at, say, the de Sitter origin point $O_{AdS} = (0, \vec{0}, \varkappa^{-1})$, since then $M_{AdS} \ni x \approx_{\varkappa\to 0} (t, \vec{r} = r\vec{n}, \varkappa^{-1})$ from equation (7)
- $\lim_{\varkappa\to 0} Sp(4,\mathbb{R}) = \mathcal{P}_+^\uparrow(1,3) = M_0 \rtimes SL(2,\mathbb{C})$.

Like above, the ten de Sitter Killing vectors (14) contract to their Poincaré counterparts $K_{\mu\nu}, \Pi_\mu, \mu = 0,1,2,3$, after rescaling the four $K_{5\mu} \longrightarrow \Pi_\mu = \varkappa K_{5\mu}$.

4.1. *Contraction limits de Sitter → Minkowski*

We have to distinguish between the Poincaré massive and massless cases. We shall denote by $\mathcal{P}^{\gtrless}(m,s)$ the positive (resp. negative) energy Wigner UIR's of the Poincaré group with mass m and spin s. We here insist again on the non-ambiguous definition of Minkowskian mass through the mass m label of a UIR of the Poincaré group. On the other hand, the notion of mass in 'de Sitterian Physics' is source of confusion. An interesting discussion and proposal on this matter is found in the work by Garidi [8], in which the following 'mass' formula has been given in terms of the dS RUI parameters p and q:

$$m_H^2 = \left(\langle Q^{(1)}\rangle_{\rm dS} - \left\langle Q^{(1)}_{p=q}\right\rangle_{\rm dS}\right)\hbar^2 H^2/c^4 = [(p-q)(p+q-1)]\hbar^2 H^2/c^4. \tag{19}$$

This formula is natural in the sense that when the second-order wave equation

$$(Q^{(1)} - \langle Q^{(1)}\rangle_{\rm dS})\varphi = 0, \tag{20}$$

obeyed by rank r tensor fields carrying a dS UIR, is written in terms of the Laplace–Beltrami operator $\Box_H$ on the dS manifold, one gets (in units $\hbar = 1 = c$)

$$(\Box_H + H^2 r(r+2) + H^2\langle Q^{(1)}\rangle_{\rm dS})\varphi = 0. \tag{21}$$

Moreover, the minimal value assumed by the eigenvalues of the first Casimir in the set of RUI in the discrete series is precisely reached at $p = q$, which corresponds to the 'conformal' massless case, as will be shown below. The Garidi mass has the advantages to encompass all mass formulae introduced within a de Sitterian context, often in a purely mimetic way in regard with their Minkowskian counterparts.

Whenever $\langle Q^{(1)}\rangle$ does not correspond to a UIR with unambiguous Minkowskian interpretation, one can still use m_H^2 but without referring to a Minkowskian meaning.

For the Poincaré massless case we shall make use of similar notation $\mathcal{P}^{\gtrless}(0, s)$ where s reads for helicity. In the latter case, conformal invariance leads us to deal also with the discrete series representations (and their lower limits) of the (universal covering of the) conformal group or its double covering $SO_0(2, 4)$ or its fourth covering $SU(2, 2)$. These UIR's are denoted in the following by $\mathcal{C}^{\gtrless}(\varsigma, j_1, j_2)$, where $(j_1, j_2) \in \mathbb{N}/2 \times \mathbb{N}/2$ labels the UIR's of the $SU(2) \times SU(2)$ subgroup and ς stems for the positive (resp. negative) conformal energy. The de Sitter contraction limits are summarized in diagrams below.

dS massive case. Solely the principal series representations are involved here (from where the name of de Sitter 'massive representations'). Introducing the representation parameter $\nu \in \mathbb{R}$ through $q = \frac{1}{2} + \mathrm{i}\nu$ or equivalently $\sigma = \nu^2 + 1/4$ (note that dS UIR corresponding to ν and $-\nu$ are equivalent) and for a spin s, the Casimir eigenvalue and Garidi mass read respectively:

$$\langle Q^{(1)}\rangle_{\rm dS} = -s(s+1) + \nu^2 + \frac{9}{4}, \tag{22}$$

$$m_H = \frac{\hbar H}{c^2}\sqrt{\left(s - \frac{1}{2}\right)^2 + \nu^2}. \tag{23}$$

Let m be a mass in the Poincaré–Minkowski sense defined by

$$m = |\nu|\hbar H/c^2 = |\nu|\varkappa\frac{\hbar}{c} = |\nu|\frac{\hbar}{c}\sqrt{\frac{|\Lambda|}{3}}. \tag{24}$$

Then we have the following general result on contraction of dS principal series representations [14, 15]:

$$\Upsilon_{s,\sigma} \underset{\substack{|\nu|\varkappa = \frac{mc}{\hbar} \\ \varkappa\to 0, |\nu|\to\infty}}{\longrightarrow} c_>\mathcal{P}^>(m, s) \oplus c_<\mathcal{P}^<(m, s), \tag{25}$$

where one of the 'coefficients' among $c_<, c_>$ can be fixed to 1 whilst the other one will vanish. Note that $m = m_H + O(\varkappa)$. Note also here the evidence of the energy ambiguity in de Sitter relativity, exemplified by the possible breaking of dS irreducibility into a direct sum of two

Poincaré UIR's with positive and negative energies, respectively. This phenomenon is linked to the existence in the de Sitter group of a specific discrete symmetry, precisely $\gamma_0 \in Sp(2,2)$, which sends any point $(x^0, \mathcal{P}) \in M_{\mathrm{dS}}$ (with the notations of (2.7)) into its mirror image $(x^0, -\mathcal{P}) \in M_{\mathrm{dS}}$ with respect to the x^0-axis. Under such a symmetry the four generators L_{a0}, $a = 1, 2, 3, 4$ (and particularly L_{40} which contracts to energy operator!) transform into their respective opposite $-L_{a0}$, whereas the six L_{ab}'s remain unchanged.

dS massless (conformal) case. Here we have $m_H = 0$ for all involved representations. Now, we must distinguish between the scalar massless case, which involves the unique complementary series UIR $\Upsilon_{0,0}$ (for which $\langle Q^{(1)}\rangle_{\mathrm{dS}} = 2$) to be contractively Poincaré significant, and the spinorial case where are involved all representations $\Pi^{\pm}_{s,s}$, $s > 0$ for which $\langle Q^{(1)}\rangle_{\mathrm{dS}} = -2(s^2 - 1)$ and lying at the lower limit of the discrete series. The arrows $\hookrightarrow$ below designate unique extension.

dS scalar massless case

$$\Upsilon_{0,0} \hookrightarrow \begin{array}{c} \mathcal{C}^{>}(1,0,0) \\ \oplus \\ \mathcal{C}^{<}(-1,0,0) \end{array} \xrightarrow{\varkappa=0} \begin{array}{c} \mathcal{C}^{>}(1,0,0) \\ \oplus \\ \mathcal{C}^{<}(-1,0,0) \end{array} \begin{array}{c} \hookleftarrow \\ \\ \hookleftarrow \end{array} \begin{array}{c} \mathcal{P}^{>}(0,0) \\ \oplus \\ \mathcal{P}^{<}(0,0), \end{array} \tag{26}$$

dS spinorial massless case

$$\Pi^{+}_{s,s} \hookrightarrow \begin{array}{c} \mathcal{C}^{>}(s+1,s,0) \\ \oplus \\ \mathcal{C}^{<}(-s-1,s,0) \end{array} \xrightarrow{\varkappa=0} \begin{array}{c} \mathcal{C}^{>}(s+1,s,0) \\ \oplus \\ \mathcal{C}^{<}(-s-1,s,0) \end{array} \begin{array}{c} \hookleftarrow \\ \\ \hookleftarrow \end{array} \begin{array}{c} \mathcal{P}^{>}(0,s) \\ \oplus \\ \mathcal{P}^{<}(0,s), \end{array} \tag{27}$$

$$\Pi^{-}_{s,s} \hookrightarrow \begin{array}{c} \mathcal{C}^{>}(s+1,0,s) \\ \oplus \\ \mathcal{C}^{<}(-s-1,0,s) \end{array} \xrightarrow{\varkappa=0} \begin{array}{c} \mathcal{C}^{>}(s+1,0,s) \\ \oplus \\ \mathcal{C}^{<}(-s-1,0,s) \end{array} \begin{array}{c} \hookleftarrow \\ \\ \hookleftarrow \end{array} \begin{array}{c} \mathcal{P}^{>}(0,-s) \\ \oplus \\ \mathcal{P}^{<}(0,-s). \end{array} \tag{28}$$

Finally, all other representations have either a non-physical Poincaré contraction limit or have no contraction limit at all. In particular, we have for the so-called *massless minimally coupled field* which corresponds to the UIR $\Pi^{+}_{1,0}$ lying at the lowest limit of the discrete series the following values for Casimir eigenvalue and Garidi mass:

$$\langle Q^{(1)}\rangle_{\mathrm{dS}} = 0, m_H = 0. \tag{29}$$

This representation, and hence the corresponding field, is exceptional under many aspects. First, it is the only one among all non-massless dS representations for which the Garidi mass vanishes, and it is part of an indecomposable structure issued from the existence of (constant) gauge solutions to (29). Secondly, it has been playing a crucial role in inflation theories, it is part of the Gupta–Bleuler structure for the massless spin-1 dS field (de Sitter QED) described by the UIR's $\Pi^{+}_{1,1}$ [17], and it is the elementary brick for the construction of the massless spin-2 dS fields (de Sitter linear gravity) described by the UIR's $\Pi^{+}_{2,2}$ [18]. Finally, the corresponding covariant quantum field theory requires a specific treatment due precisely to its indecomposable nature [19].

4.2. Contraction limits anti de Sitter → Minkowski

A 'mass' formula analogous to the Garidi one can be proposed here in the case of the AdS discrete series. It will give a zero-mass for massless AdS fields:

$$m_H^2 = \left(\langle Q^{(1)}\rangle_{\mathrm{AdS}} - \left\langle Q^{(1)}_{\varsigma=s+1}\right\rangle_{\mathrm{AdS}}\right)\hbar^2 H^2/c^4,$$
$$\text{i.e.}\, m_H = \frac{\hbar H}{c^2}\sqrt{\left(\varsigma - \frac{3}{2}\right)^2 - \left(s - \frac{1}{2}\right)^2}. \tag{30}$$

With the same notations as above, the anti de Sitter contraction limits can be summarized in the following diagrams.

AdS massive case. Solely the (holomorphic) discrete series representations $D(\varsigma, s)$ with $\varsigma > s + 1$ are involved here. Introducing the following relation between the representation parameter $\varsigma > s + 1$ and a Poincaré–Minkowski mass:

$$m = \varsigma\varkappa\frac{\hbar}{c} = \varsigma\frac{\hbar}{c}\sqrt{\frac{|\Lambda|}{3}}, \tag{31}$$

we have [16]

$$D(\varsigma, s) \underset{\substack{\varsigma\varkappa = \frac{mc}{\hbar} \\ \varkappa\to 0, \varsigma\to\infty}}{\longrightarrow} \mathcal{P}^{>}(m, s). \tag{32}$$

Note here that there is no energy ambiguity in anti de Sitter relativity (there are other ambiguities!). If we wished to get the negative energy Poincaré representations, we would instead have chosen the representations in the *antiholomorphic* discrete series (in which the spectrum of the compact generator L_{50} is bounded above by $-\varsigma$, $\varsigma > 0$):

$$D(-\varsigma, s) \underset{\varkappa\to 0, \varsigma\to\infty}{\longrightarrow} \mathcal{P}^{<}(m, s). \tag{33}$$

AdS massless (conformal) case Now we must distinguish between the scalar massless case, which involves the UIR $D(1, 0)$ and the spinorial–tensorial case in which are involved all representations $D(s + 1, s)$, $s > 0$ lying at the lower limit of the holomorphic discrete series. Here, there is no ambiguity concerning energy, but there is ambiguity concerning helicity, since the later is not defined in AdS. As above, the arrows $\hookrightarrow$ below designate unique extension.*AdS scalar massless case*

$$D(1, 0) \quad \hookrightarrow \quad \mathcal{C}^{>}(1, 0, 0) \quad \overset{\varkappa=0}{\longrightarrow} \quad \mathcal{C}^{>}(1, 0, 0) \quad \hookleftarrow \quad \mathcal{P}^{>}(0, 0). \tag{34}$$

AdS spinorial tensorial massless case

$$D(s+1, s) \quad \hookrightarrow \quad \begin{array}{c} \mathcal{C}^{>}(s+1, s, 0) \\ \oplus \\ \mathcal{C}^{>}(s+1, 0, s) \end{array} \quad \overset{\varkappa=0}{\longrightarrow} \quad \begin{array}{c} \mathcal{C}^{>}(s+1, s, 0) \\ \oplus \\ \mathcal{C}^{>}(s+1, 0, s) \end{array} \quad \begin{array}{c} \hookleftarrow \\ \\ \hookleftarrow \end{array} \quad \begin{array}{c} \mathcal{P}^{>}(0, s) \\ \oplus \\ \mathcal{P}^{>}(0, -s). \end{array} \tag{35}$$

Finally, all other representations have either a non-physical Poincaré contraction limit or have no contraction limit at all. In particular, we have for the *Rac* and *Di* fields the following respective values for Casimir eigenvalue and Garidi mass:

$$\langle Q^{(1)}\rangle_{\mathrm{AdS}} = -\frac{5}{4}, \qquad m_H = \frac{\sqrt{3}}{2}\frac{\hbar H}{c^2}(\mathrm{Rac}), \tag{36}$$

$$\langle Q^{(1)}\rangle_{\mathrm{AdS}} = -\frac{5}{4}, \qquad m_H = \frac{\hbar H}{2c^2}(\mathrm{Di}). \tag{37}$$

It should also be noted that, like for de Sitter, there exists a unique UIR, among all non-massless AdS representations, for which m_H vanishes, namely the UIR $D(2, 0)$ in the discrete series.

5. Contraction limits and the question of interpretation of mass in dS and AdS spacetimes

Actually, both contraction formulae (24) and (31), established on a group irrep. level, are by far restrictive. Of course, they give the abstract dimensionless parameters ν and ς labeling respectively the UIR's of the dS and AdS groups a status of physical quantity in terms of measurable other physical quantities, like a mass m and a cosmological constant Λ (universal?), and of universal constants, like c and $\hbar$. However, given a Minkowskian mass m and a 'universal' length $R = \varkappa^{-1} =: \sqrt{3/|\Lambda|} = cH^{-1}$, nothing prevents us from considering those two quantities, specific of a 'physics' in constant-curvature spacetime, as meromorphic functions of the dimensionless physical quantity

$$\vartheta \equiv \vartheta_m \stackrel{\text{def}}{=} \frac{\hbar\varkappa}{mc} = \frac{\hbar}{Rmc} = \frac{\hbar\sqrt{|\Lambda|}}{\sqrt{3}mc} = \frac{\hbar H}{mc^2}. \tag{38}$$

Note that this quantity is also the ratio of the Compton length of the Minkowskian object of mass m considered at the limit with the universal length $R = \varkappa^{-1}$ yielded by dS or AdS geometry. It reduces to $\varkappa/m$ in units $\hbar = 1 = c$.

Now, we may consider the following Laurent expansions of ν (for the dS principal series) and ς (for the AdS discrete series) in a certain neighborhood of $\vartheta = 0$:

$$\nu = \nu(\vartheta) = \frac{1}{\vartheta} + e_0 + e_1\vartheta + \cdots + e_n\vartheta^n + \cdots \tag{39}$$

$$\varsigma = \varsigma(\vartheta) = \frac{1}{\vartheta} + f_0 + f_1\vartheta + \cdots + f_n\vartheta^n + \cdots, \qquad \vartheta \in (0, \vartheta_1) \text{ convergence interval}, \tag{40}$$

where the e_n, f_n are pure numbers to be determined. We should be aware that nothing is changed in the contraction formulae (25) and (32) from the point of view of a Minkowskian observer, except that we allow to consider positive as well as negative values of ν in a (positive) neighborhood of $\vartheta = 0$. By multiplying (39) and (40) by ϑ and taking the limit $\vartheta \to 0$ we recover asymptotically relations (24) and (31).

As a matter of fact, the Garidi mass (23) in the dS case or the mass formula (30) proposed for the AdS case are perfect examples of such expansions since they can be rewritten as the following expansions in the parameter $\vartheta \in (0, 1/|s - 1/2|]$:

$$\nu = \sqrt{\frac{1}{\vartheta^2} - (s - 1/2)^2} = \frac{1}{\vartheta} - (s - 1/2)^2 \left(\frac{\vartheta}{2} + O(\vartheta^2)\right), \tag{41}$$

$$\varsigma = \frac{3}{2} + \sqrt{\frac{1}{\vartheta^2} + (s - 1/2)^2} = \frac{1}{\vartheta} + \frac{3}{2} + (s - 1/2)^2 \left(\frac{\vartheta}{2} + O(\vartheta^2)\right). \tag{42}$$

Note the particular symmetric place occupied by the spin-1/2 case with regard to the scalar case $s = 0$ and the boson case $s = 1$.

Hence, we can tell something more on the number f_0 introduced for the anti de Sitter case, and this represents one more motivation for exploring further the possibilities offered by the above expansions. An AdS scalar elementary system can be viewed as a deformation of both a relativistic free particle with rest energy mc^2 and a harmonic oscillator with rest energy $\frac{3}{2}\hbar\omega$, with frequency $\omega = H^{-1}$. The following has thus been proven in [20] in the (1 + 1)-dimensional case:

$$\varsigma = \frac{mc}{\hbar\varkappa} + \frac{1}{2} + O(\varkappa), \tag{43}$$

which means precisely that $f_0 = 1/2$ in this case from which is derived the following expansion of the energy of a scalar 'massive' AdS elementary system from a Minkowskian tangent point of view:

$$E_{\mathrm{AdS}} = mc^2 + \tfrac{1}{2}\hbar\omega + O(\varkappa). \tag{44}$$

The extension of the proof to the (3 + 1)-dimensional case is straightforward and the result is in perfect agreement with the content of the expansion (42) concerning the appearance of the constant term 3/2:

$$E_{\mathrm{AdS}} = mc^2 + \tfrac{3}{2}\hbar\omega + O(\varkappa). \tag{45}$$

It is an amazing feature of AdS to reveal universal pure vibration energy besides matter energy.

On the other hand, the situation of dS relativity is less tractable. It is well exemplified by the absence of any constant term in (41).

Let us insist once more on the very peculiar position occupied by the spin $s = 1/2$ since then we exactly have from (41) and (42):

$$\nu = \frac{1}{\vartheta} \text{and} E_{\mathrm{AdS}} = mc^2 + \frac{3}{2}\hbar\omega. \tag{46}$$

So, in this particular case, the range of possible values for ϑ is the positive real axis $(0, \infty)$.

6. Conclusion

In this paper, we have discussed about giving a basic physical quantity like mass an asymptotic meaning in the presence of a constant curvature background. Actually, we face two possibilities: either one starts from a Minkowskian background, and turn on gravity (in particular in order to get the (anti) de Sitter structure) or one starts directly from within the framework of a (anti) de Sitter geometry. In the first case, an arbitrary field has a well-defined mass due to Poincaré invariance. In the second case, one is forced to adopt another invariant for describing the free field. We have proposed here to follow the idea of Garidi in defining a mass in the dS/AdS sense which has a consistent flat limit.

The current observation of an accelerated universe points in favor of a desitterian arena. In consistency with this fact, we propose to reexamine carefully the question of mass, or at least accept the existence of two points of view on this matter. Ignoring one of them by preferring the other one could lead to serious misleading in the interpretations of experiments or observations.

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